

UNIT DISTANCES AND DIAMETERS IN EUCLIDEAN SPACES

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ABSTRACT. We show that the maximum number of unit distances or of diameters in a set of n points in d -dimensional Euclidean space is attained only by specific types of Lenz constructions, for all $d \geq 4$ and n sufficiently large, depending on d . As a corollary we determine the exact maximum number of unit distances for all even $d \geq 6$, and the exact maximum number of diameters for all $d \geq 4$, for all n sufficiently large, depending on d .

1. INTRODUCTION

1.1. Unit distances. For a finite subset S of Euclidean d -space \mathbb{R}^d let $u(S)$ denote the number of pairs of points in S at distance 1. Define

$$u_d(n) = \max\{u(S) : S \subset \mathbb{R}^d, |S| = n\}.$$

Erdős initiated the study of $u_2(n)$ in [6] and the higher-dimensional case of $u_d(n)$, $d \geq 3$, in [7]. The cases $d = 2$ and $d = 3$ are the most difficult. Erdős [6] obtained the superlinear lower bound

$$u_2(n) \geq n^{1 + \frac{c}{\log \log n}},$$

which he conjectured to be tight [11, 12, 13, 14]. The best known upper bound is

$$u_2(n) \leq cn^{4/3},$$

due to Spencer, Szemerédi and Trotter [28]. See Székely [33] for a particularly simple proof.

For $d = 3$ the known lower (Erdős [7]) and upper bounds (Clarkson et al. [5]) are:

$$cn^{4/3} \log \log n \leq u_3(n) \leq cn^{3/2} \beta(n),$$

where $\beta(n)$ is an extremely slowly growing function related to the inverse Ackerman function.

For $d \geq 4$ (the subject of this paper) the situation changes drastically. Lenz, as reported in [7], observed that if we take $p := \lfloor d/2 \rfloor$ circles in pairwise orthogonal 2-dimensional subspaces, each with centre the origin and radius $1/\sqrt{2}$, then any two points on different circles are at unit distance. Therefore, if n points are chosen by taking $n/p + O(1)$ points on each circle, $\frac{p-1}{2p}n^2 - O(1)$ unit distances are obtained. Erdős [7] showed that since $K_{p+1}(3)$, the complete $(p+1)$ -partite graph with three vertices in each class,

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does not occur as a unit-distance graph in \mathbb{R}^d , the Erdős-Stone theorem gives:

$$u_d(n) = \frac{p-1}{2p}n^2 + o(n^2) \text{ for all } d \geq 4.$$

Using an extremal graph theory result of Erdős [8] and Simonovits [27], Erdős [9] determined the exact value of $u_d(n)$ for d even and n a sufficiently large (depending on d) multiple of $2d = 4p$. The n/p points on each circle are then taken to be the vertices of $n/(4p)$ squares. This determines $u_d(n)$ asymptotically for all sufficiently large n up to a $O(1)$ term (still for d even). Brass [1], together with a number theoretical result of Van Wamelen [34], determined $u_4(n)$ completely. For $n \geq 5$,

$$u_4(n) = \begin{cases} \lfloor n^2/4 \rfloor + n & \text{if } n \text{ is divisible by 8 or 10,} \\ \lfloor n^2/4 \rfloor + n - 1 & \text{otherwise.} \end{cases}$$

For odd $d \geq 5$ Erdős and Pach [16] showed that

$$u_d(n) = \frac{p-1}{2p}n^2 + \Theta(n^{4/3}).$$

For the lower bound they observed that the Lenz construction can be improved when d is odd by replacing one of the circles by a 2-sphere of radius $1/\sqrt{2}$ in a 3-dimensional space orthogonal to the other 2-dimensional subspaces and by placing the points on the sphere such that the unit distance occurs at least $cn^{4/3}$ times (a construction of Erdős, Hickerson and Pach [15]). For the upper bound they used a stability result in extremal graph theory [2, Chapter 5, remark 4.5(ii)] together with the fact that the maximum number of unit distances among n points on a 2-sphere is $O(n^{4/3})$ [5].

1.2. Diameters. For a finite subset S of \mathbb{R}^d we call a pair of points in S a *diameter* if their distance equals the diameter of S . Let $M(S)$ denote the number of diameters in S . Define

$$M_d(n) = \max\{M(S) : S \subset \mathbb{R}^d, |S| = n\}.$$

Erdős in [6] showed that $M_2(n) = n$ for $n \geq 3$. Vázsonyi conjectured, as reported in [6], that $M_3(n) = 2n - 2$ for $n \geq 4$. This was independently proved by Grünbaum [19], Heppes [20] and Straszewicz [29]. For a new proof, see [31].

As in the case of unit distances, the situation is completely different when $d \geq 4$. Erdős [7] showed that for $d \geq 4$, $M_d(n) = \frac{p-1}{2p}n^2 + o(n^2)$, the same asymptotics as $u_d(n)$. For other work on this problem by Hadwiger, Lenz and Yagai, see the survey of Martini and Soltan [24].

2. NEW RESULTS

If $d \geq 4$ is even, let $p = d/2$ and consider any orthogonal decomposition $\mathbb{R}^d = V_1 \oplus \dots \oplus V_p$, where each V_i is 2-dimensional. In each V_i , let C_i be the circle with centre the origin o and radius r_i such that $r_i^2 + r_j^2 = 1$ for all distinct i and j . When $d \geq 6$ this implies that each $r_i = 1/\sqrt{2}$. We define a *Lenz configuration* to be any translate of a finite subset of $\bigcup_{i=1}^p C_i$.

If $d \geq 5$ is odd, let $p = \lfloor d/2 \rfloor$, and consider any orthogonal decomposition $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$, where V_1 is 3-dimensional and each V_i ($i = 2, \dots, p$) is 2-dimensional. Let Σ be the sphere in V_1 with centre o and radius r_1 , and for each $i = 2, \dots, p$, let C_i be the circle with centre o and radius r_i , such that $r_i^2 + r_j^2 = 1$ for all distinct i, j . When $d \geq 7$, necessarily each $r_i = 1/\sqrt{2}$. We define a *Lenz configuration* to be any translate of a finite subset of $\Sigma \cup \bigcup_{i=2}^p C_i$. (Later we distinguish between *weak* and *strong* Lenz configurations as a technical notion inside the proofs. The definition here coincides with a strong Lenz construction in the sequel).

We call a set S of n points in \mathbb{R}^d an *extremal set* with respect to unit distances [diameters] if $u(S) = u_d(n)$ [$M(S) = M_d(n)$].

Theorem 1. *For each $d \geq 4$ there exists $N(d)$ such that all extremal sets of $n \geq N(d)$ points (with respect to unit distances or diameters) are Lenz configurations.*

The proof uses a typical technique in extremal graph and hypergraph theory [27, 18, 22, 25]: First prove a stability result for sets that are close to extremal, and then deduce more exact structural information from extremality.

For even $d \geq 6$ it is then possible to determine $u_d(n)$ exactly. On the other hand, for odd $d \geq 5$ the main obstacle to determine $u_d(n)$ is our lack of knowledge of the function $f(m)$ which gives the exact maximum number of unit distances between m points on a 2-sphere of radius $1/\sqrt{2}$ (for odd $d \geq 7$) and the function $g(m)$ which gives the exact maximum number of unit distances between m points on a sphere of arbitrary radius [15, 32] (for $d = 5$).

Let $t_p(n)$ denote the number of edges of the *Turán p -partite graph* on n vertices. This is the complete p -partite graph with $\lfloor n/p \rfloor$ or $\lceil n/p \rceil$ in each class [2, Chapter VI]. We do not need the exact value of $t_p(n)$, only that

$$t_p(n) = \frac{p-1}{2p}n^2 - O(1).$$

Corollary 2. *Let $d \geq 6$ be even. For all sufficiently large n (depending on d),*

$$u_d(n) = \begin{cases} t_p(n) + n - r & \text{if } 0 \leq r \leq p-1, \\ t_p(n) + n - p & \text{if } p \leq r \leq 3p-1, \\ t_p(n) + n - 2d + r & \text{if } 3p \leq r \leq 4p-1, \end{cases}$$

where $p = d/2$ and r is the remainder when dividing n by $4p = 2d$.

For all $d \geq 4$ it is possible to determine $M_d(n)$ exactly if n is large. The most complicated case is $d = 5$, where it is necessary to know the maximum number of diameters in a set of n points on a 2-sphere in \mathbb{R}^3 . For each $n \geq 6$ we construct a set of n points in \mathbb{R}^3 with $2n - 2$ diameters, all lying on a sphere (see Lemma 7(e) below).

Corollary 3. *For all sufficiently large n (depending on d),*

$$\begin{aligned} M_4(n) &= \begin{cases} t_2(n) + \lceil n/2 \rceil + 1 & \text{if } n \not\equiv 3 \pmod{4}, \\ t_2(n) + \lceil n/2 \rceil & \text{if } n \equiv 3 \pmod{4}; \end{cases} \\ M_5(n) &= t_2(n) + n; \\ M_d(n) &= t_p(n) + p \quad \text{for even } d \geq 6, \text{ where } p = d/2; \\ M_d(n) &= t_p(n) + \lceil n/p \rceil + p - 1 \quad \text{for odd } d \geq 7, \text{ where } p = \lfloor d/2 \rfloor. \end{aligned}$$

We use two stability theorems to prove Theorem 1, one for even dimensions and one for odd dimensions.

Theorem 4. *For each $\varepsilon > 0$ and even $d \geq 4$ there exist $\delta > 0$ and N such that any set of $n \geq N$ points in \mathbb{R}^d with at least $(\frac{p-1}{2p} - \delta)n^2$ unit distance pairs can be partitioned into S_0, S_1, \dots, S_p such that $|S_0| < \varepsilon n$ and for each $i = 1, \dots, p$,*

$$\frac{n}{p} - \varepsilon n < |S_i| < \frac{n}{p} + \varepsilon n$$

and S_i is on a circle C_i , such that the circles C_1, \dots, C_p have the same centre and are mutually orthogonal.

Theorem 5. *For each $\varepsilon > 0$ and odd $d \geq 5$ there exist $\delta > 0$ and N such that any set S of $n \geq N$ points in \mathbb{R}^d with at least $(\frac{p-1}{2p} - \delta)n^2$ unit distance pairs can be partitioned into S_0, S_1, \dots, S_p such that $|S_0| < \varepsilon n$ and for each $i = 1, \dots, p$,*

$$\frac{n}{p} - \varepsilon n < |S_i| < \frac{n}{p} + \varepsilon n,$$

S_1 is on a 2-sphere Σ_1 , S_i is on a circle C_i , $i = 2, \dots, p$, and $\Sigma_1, C_2, \dots, C_p$ have the same centre and are mutually orthogonal.

Corollary 6. *Let $d \geq 4$. If a set S of n points in \mathbb{R}^d has at least $(\frac{p-1}{2p} - o(1))n^2$ unit distance pairs, then S is a Lenz configuration except for $o(n)$ points.*

3. OVERVIEW OF THE PAPER

In Section 4 we consider results from geometry necessary for the proofs.

In Section 5 we determine the maximum number of unit distances and diameters in even-dimensional Lenz configurations, introduce the notions of weak and strong Lenz configuration in odd dimensions, show that the weak Lenz configurations with the largest number of unit distances or diameters are strong Lenz configurations, and determine the maximum number of diameters in strong Lenz configurations. Corollaries 2 and 3 then follow, given that extremal sets are (weak) Lenz configurations.

In Section 6 we use the Erdős-Simonovits stability theorem from extremal graph theory to prove Theorems 4 and 5, from which Corollary 6 is immediate.

Finally, in Section 7 we use the stability theorems to show that sets of points that are extremal with respect to unit distances or diameters are (weak) Lenz configurations, thereby finishing the proof of Theorem 1.

4. GEOMETRIC PRELIMINARIES

We denote the distance between points p and q in \mathbb{R}^d by $|pq|$. The *unit distance graph* of a set S of n points in \mathbb{R}^d is defined by joining any two points at distance 1. Let $u(S)$ denote the number of (unordered) unit distance pairs in S . Two points in S at distance 1 are *neighbours*. For any point x and finite set S , let $u(x, S)$ denote the number of points in A that are at distance 1 to x . Similarly, for any finite sets A and B , let $u(A, B)$ denote the number of (ordered) unit distance pairs (a, b) with $a \in A$ and $b \in B$.

Whenever we work with diameters, we assume that the diameter of S is 1, and then we use the notation $u(S)$, $u(x, S)$ and $u(A, B)$ as before. In this case we call the unit distance graph of S the *diameter graph* of S .

We continually use the following two basic lemmas in the sequel. The first deals with unit distances and diameters on circles and 2-spheres, and the second with unit distances in dimensions higher than 3.

Lemma 7. *Let S be a set of n points in \mathbb{R}^3 .*

(a) *If S lies on a circle of radius $1/\sqrt{2}$, then*

$$u(S) \leq \begin{cases} n & \text{if } n \text{ is divisible by 4,} \\ n-1 & \text{otherwise.} \end{cases}$$

Equality is possible for all n , by letting S be the union of the vertices of $\lfloor n/4 \rfloor$ inscribed squares and $n - 4\lfloor n/4 \rfloor$ vertices of an additional square.

(b) *If S has diameter 1 and lies on a circle, then*

$$u(S) \leq \begin{cases} n & \text{if } n \text{ is odd,} \\ n-1 & \text{if } n \text{ is even.} \end{cases}$$

Equality is possible for all $n \geq 2$, for a circle of suitable radius depending on n .

(c) *If S has diameter 1 and lies on a circle of radius $> 1/\sqrt{3}$, then $u(S) = 1$.*

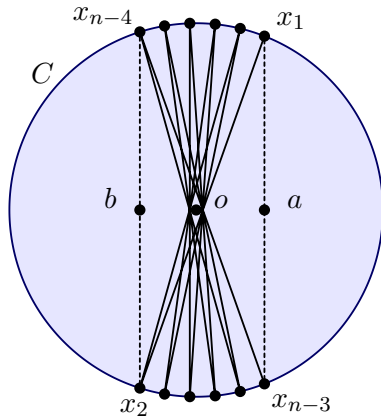
(d) *If S lies on a 2-sphere, then $u(S) = O(n^{4/3})$. There exist sets S with $u(S) = \Omega(n^{4/3})$.*

(e) *If S has diameter 1 and lies on a 2-sphere, then $u(S) \leq 2n - 2$. Equality is possible for each $n \geq 4$, $n \neq 5$, for a 2-sphere of suitable radius depending on n .*

(f) *If S has diameter 1 and lies on a 2-sphere of radius $\geq 1/\sqrt{2}$, then $u(S) \leq n$. Equality is possible for all $n \geq 3$ and all radii $\geq 1/\sqrt{2}$.*

Proof. Statements (a), (b), (c) are straightforward, except perhaps $u(S) \leq n - 1$ for an even number of concyclic points of diameter 1. This follows essentially from the easily seen observation that if the diameter graph of points of some concyclic points contains a cycle, then it consists only of this cycle, together with the well known fact that all cycles in diameter graphs in the plane are odd [21, 30].

The upper bound in (d) is due to Clarkson et al. [5]. The simplest known proof of it is by adapting Székely's proof [33] for the planar case. The lower bound in (d) is due to Erdős, Hickerson and Pach [15].



Statement (f) can be found in Kupitz, Martini and Wegner [23]. It follows as in the planar case [26, Theorem 13.13] from the observation that any two diameters, when drawn as short great circular arcs on the 2-sphere, must intersect. Examples of n points with n diameters are easily found for all radii larger than $1/\sqrt{2}$; they have essentially the same structure as in the plane; see [23] for details.

The only statement that remains to be proved, is that $2n - 2$ diameters can be attained on a 2-sphere for each $n \geq 4$, $n \neq 5$. For even $n \geq 4$ the construction is easy. Consider the vertex set of a regular $(n - 1)$ -gon of diameter 1, and choose another point on the axis of symmetry of the polygon at distance 1 to the $n - 1$ vertices. This clearly gives n points with $2n - 2$ diameters.

For odd $n \geq 7$ the construction is more involved. Place $n - 3$ points x_1, \dots, x_{n-3} on the circle C of radius r and centre o in the xy -plane such that the diameter 1 occurs between consecutive x_i 's (Figure 1). Note that r

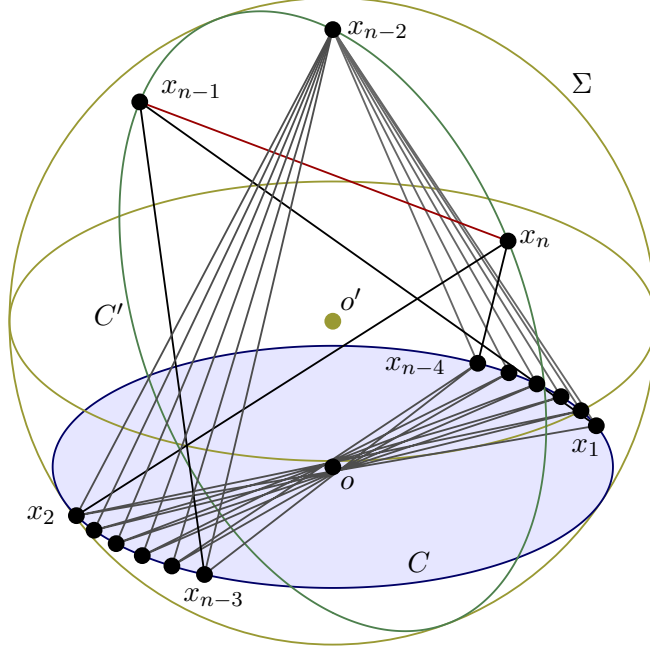


FIGURE 2. 15 points on a sphere with 28 diameters

and n determine everything up to isometry. We fix r later in the proof. Let x_{n-2} be the point on the positive z -axis at distance 1 to C . Then x_{n-2} and C are on a unique sphere Σ with centre o' and radius s , say. Note that o' is on the positive z -axis.

We now want to find points x_{n-1} and x_n on Σ such that

$$|x_1x_{n-1}| = |x_{n-3}x_{n-1}| = |x_2x_n| = |x_{n-4}x_n| = |x_{n-1}x_n| = 1$$

and

$$|x_{n-2}x_{n-1}| \leq 1, \quad |x_{n-2}x_n| \leq 1.$$

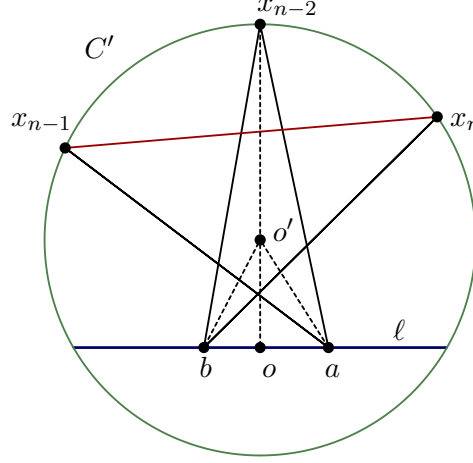
See Figure 2. This will give the required number of diameters in the set $S := \{x_1, \dots, x_n\}$. For any value of r there will clearly be unique points $x_{n-1}, x_n \in \Sigma \setminus \{x_{n-2}\}$ that satisfy

$$|x_{n-3}x_{n-1}| = |x_1x_{n-1}| = |x_2x_n| = |x_{n-4}x_n| = 1.$$

It remains to find an appropriate value of r so that

$$|x_{n-1}x_n| = 1, \quad |x_{n-2}x_{n-1}| \leq 1, \quad |x_{n-2}x_n| \leq 1.$$

We reduce this to a two-dimensional problem. Let a and b be the mid-points of x_1x_{n-3} and x_2x_{n-4} , respectively. Consider the intersection of Σ with the plane $oabx_{n-2}$. This is a circle C' with centre o' and radius s . By symmetry, x_{n-1} and x_n lie on C' , and $|ax_{n-2}| = |ax_{n-1}|$ and $|bx_{n-2}| = |bx_n|$ (Figure 3). Therefore, ao' bisects $\angle x_{n-2}ax_{n-1}$, and bo' bisects $\angle x_{n-2}bx_{n-1}$.

FIGURE 3. Circle C'

Clearly, $|oa| > |ob|$, and both $|oa|$ and $|ob|$ are strictly monotone functions of r .

We now consider r to be a variable ranging in the interval $(1/2, r_0)$, where

$$r_0 := \left(2 \cos \frac{\pi}{2(n-4)} \right)^{-1}.$$

On the one hand $r > \frac{1}{2}$, and in the limit as $r \rightarrow \frac{1}{2}$, the diameters $x_i x_{i+1}$ all coincide, and $\lim_{r \rightarrow 1/2} |oa| = \lim_{r \rightarrow 1/2} |ob| = 0$. It follows that

$$\lim_{r \rightarrow 1/2} |x_{n-2} x_{n-1}| = \lim_{r \rightarrow 1/2} |x_{n-2} x_n| = 0,$$

hence $\lim_{r \rightarrow 1/2} |x_{n-1} x_n| = 0$.

On the other hand, $r < r_0$, where in the limit as $r \rightarrow r_0$, x_1 and x_{n-3} coincide, and the points form the vertex set of a regular $(n-4)$ -gon. Thus

$$\lim_{r \rightarrow r_0} |oa| = r_0,$$

$$\lim_{r \rightarrow r_0} |ob| \rightarrow 2r_0 \sin \frac{\pi}{n-4},$$

and

$$\lim_{r \rightarrow r_0} |x_{n-2} a| = 1.$$

Since $2r_0 > 1$, $\lim_{r \rightarrow r_0} x_{n-1}$ is a point below the chord ℓ of C' through a and b . (Note that ℓ is a diameter of C). Also,

$$\lim_{r \rightarrow r_0} |x_2 a| = \lim_{r \rightarrow r_0} |x_{n-4} a| = 1,$$

hence $\lim_{r \rightarrow r_0} x_n = a$. Since x_{n-1} is lower than x_n (because $|oa| > |ob|$), when x_{n-1} reaches ℓ , x_n has not reached ℓ yet. Since $|x_n b| = |x_{n-2} b|$, it follows that the chord $x_n b$ is below o' . Since at this stage (with $x_{n-1} \in \ell$) the chord bx_{n-1} is below o' , it follows that the chord $x_{n-1} x_n$ is below o' . Thus before x_{n-1} reaches ℓ , there is a stage where $x_{n-1} x_n$ passes through o' with

both x_{n-1} and x_n still above ℓ , and therefore at distance at most 1 to x_{n-2} . From $s > r > 1$ it follows that $|x_{n-1}x_n| > 1$. Since $\lim_{r \rightarrow 1/2} |x_{n-1}x_n| = 0$, at some stage $|x_{n-1}x_n| < 1$. Therefore, at some inbetween stage, $|x_{n-1}x_n| = 1$. This finishes the construction for odd $n \geq 7$. \square

We remark that the exception $n \neq 5$ in Lemma 7(f) is necessary. Suppose there exist 5 points on a 2-sphere with 8 diameters. Then one of the points must be incident to 4 diameters. The other 4 points are then concyclic, and among them there can be at most 3 diameters (Lemma 7(b)), a contradiction. On the other hand, it is easy to find 5 points on a sphere with 7 diameters.

The next lemma is well known. We omit the easy proof.

Lemma 8. *Let A and B be finite subsets of \mathbb{R}^d , each of size at least 3. If $|ab| = 1$ for all $a \in A$, $b \in B$, then the affine subspaces spanned by A and B are orthogonal, A and B lie on spheres of radii r_a and r_b , say, such that $r_a^2 + r_b^2 = 1$, and with common centre the point of intersection of the two subspaces.*

5. OPTIMISED LENZ CONFIGURATIONS

5.1. Even dimensions $d \geq 6$. We have already defined a Lenz configuration in the introduction. For any Lenz configuration S on n points lying on $p = d/2$ mutually orthogonal circles C_i with centre o and radius $1/\sqrt{2}$, we define $S_i := S \cap C_i$ and $n_i := |S_i|$.

5.1.1. Unit distances. Define

$$u_d^L(n) = \max\{u(S) : S \text{ is a Lenz configuration of } n \text{ points in } \mathbb{R}^d\}.$$

We call any Lenz configuration S of n points in \mathbb{R}^d for which $u(S) = u_d^L(n)$ an *optimised Lenz configuration* (for unit distances).

Proposition 9. *Let $d \geq 6$ be even, $n \geq 1$, $p = d/2$ and $n \equiv r \pmod{2d}$, $0 \leq r \leq 2d - 1$. Then*

$$u_d^L(n) = \begin{cases} t_p(n) + n - r & \text{if } 0 \leq r \leq p - 1, \\ t_p(n) + n - p & \text{if } p \leq r \leq 3p - 1, \\ t_p(n) + n - 2d + r & \text{if } 3p \leq r \leq 4p - 1, \end{cases}$$

Proof. Consider an optimised Lenz configuration S on p pairwise orthogonal circles C_1, \dots, C_p . We may rearrange the points on each circle without changing the number of unit distances between circles. By Lemma 7(a) and maximality, each $u(S_i) = n_i$ if $n_i \equiv 0 \pmod{4}$ and $u(S_i) = n_i - 1$ otherwise. The problem is now that of maximising the function

$$u(n_1, \dots, n_p) := \sum_{1 \leq i < j \leq p} n_i n_j + n - p + k(n_1, \dots, n_p),$$

over all nonnegative n_1, \dots, n_p that sum to n , where $k(n_1, \dots, n_p)$ equals the number of n_i divisible by 4. This easy but tedious exercise finishes the proof. \square

5.1.2. *Diameters.* Define

$$M_d^L(n) = \max\{u(S) : S \text{ is a diameter 1 Lenz configuration of } n \text{ points in } \mathbb{R}^d\}.$$

We call any diameter 1 Lenz configuration S of n points in \mathbb{R}^d for which $u(S) = M_d^L(n)$ an *optimised Lenz configuration* (for diameters).

Proposition 10. *Let $d \geq 6$ be even, $n \geq d$, and $p = d/2$. Then*

$$M_d^L = t_p(n) + p.$$

Proof. Consider an optimised Lenz configuration S of diameter 1 on p pairwise orthogonal circles C_1, \dots, C_p . By Lemma 7(c), each $u(S_i) \leq 1$. Therefore, $u(S) \leq t_p(n) + p$. Equality is clearly possible if $n \geq d$, by dividing the n points as equally as possible between the p circles, and ensuring that a diameter occurs within each S_i . \square

5.2. The dimension $d = 4$. For any Lenz configuration S on n points lying on orthogonal circles C_1 and C_2 with common centre o and radii r_1 and r_2 such that $r_1^2 + r_2^2 = 1$, define $S_i := S \cap C_i$ and $n_i := |S_i|$.

5.2.1. *Unit distances.* This section is included for the sake of completeness. Define

$$u_4^L(n) = \max\{u(S) : S \text{ is a Lenz configuration of } n \text{ points in } \mathbb{R}^4\}.$$

As shown by Brass [1] and Van Wamelen [34]:

Proposition 11. *Let $n \geq 5$. Then*

$$u_4^L(n) = \begin{cases} t_2(n) + n & \text{if } n \text{ is divisible by 8 or 10,} \\ t_2(n) + n - 1 & \text{otherwise.} \end{cases}$$

5.2.2. *Diameters.* Define

$$M_4^L(n) = \max\{u(S) : S \text{ is a diameter 1 Lenz configuration of } n \text{ points in } \mathbb{R}^4\}.$$

We call any diameter 1 Lenz configuration S of n points in \mathbb{R}^4 for which $u(S) = M_4^L(n)$ an *optimised Lenz configuration* (for diameters).

Proposition 12. *Let $n \geq 6$. Then*

$$M_4^L(n) = \begin{cases} t_2(n) + \lceil n/2 \rceil + 1 & \text{if } n \not\equiv 3 \pmod{4}, \\ t_2(n) + \lceil n/2 \rceil & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Proof. Consider an optimised Lenz configuration S of diameter 1 on pairwise orthogonal circles C_1 and C_2 . Without loss of generality $r_1 \leq r_2$. We now apply Lemma 7(b), (c). If $u(S_2) > 1$, then $r_2 \leq 1/\sqrt{3}$ and $r_1 \geq \sqrt{2/3} > r_2$, a contradiction. Therefore, $u(S_2) \leq 1$. Also, $u(S_1) \leq n_1$, and if n_1 is even, $u(S_1) \leq n_1 - 1$. It follows that

$$u(S) \leq \begin{cases} n_1 n_2 + n_1 + 1 & \text{if } n_1 \text{ is odd,} \\ n_1 n_2 + n_1 & \text{if } n_1 \text{ is even.} \end{cases}$$

By considering the four cases of n modulo 4, it is easily checked that the maximum over all nonnegative n_i with $n_1 + n_2 = n$ is as in the statement

of the theorem. For $n \geq 6$ it is also easy to see that there are configurations that attain this maximum. \square

5.3. Odd dimensions $d \geq 7$. We introduce the notion of a weak Lenz configuration. Let $d \geq 7$ be odd, $p = (d-1)/2$, and consider any orthogonal decomposition $\mathbb{R}^d = V_0 \oplus V_1 \oplus \cdots \oplus V_p$ with $\dim V_0 = 1$ and $\dim V_i = 2$ ($i = 1, \dots, p$). For each $i = 1, \dots, p$, let Σ_i be the sphere in $V_0 \oplus V_i$ with centre o and radius $1/\sqrt{2}$, and let C_i be the circle in V_i with centre o and radius $1/\sqrt{2}$. Let p^+ and p^- be the two points in V_0 at distance $1/\sqrt{2}$ from o . Then p^+ and p^- are the north and south poles of each Σ_i when C_i is considered to be its equator.

Let $i \neq j$. If some $x \in \Sigma_i$ is at unit distance to some point of $\Sigma_j \setminus C_j$, then x is at unit distance to all of Σ_j (since it is already at unit distance to C_j). By Lemma 8, $x \in C_i$. It follows that no point of $\Sigma_i \setminus C_i$ can be at unit distance to a point of $\Sigma_j \setminus C_j$.

A *strong Lenz configuration* of n points in \mathbb{R}^d is a translate of a finite subset of $C_1 \cup \cdots \cup C_{p-1} \cup \Sigma_p$ for some orthogonal decomposition. (This is merely the odd-dimensional “Lenz configuration” of Section 2.) A *weak Lenz configuration* of n points in \mathbb{R}^d is a translate of a finite subset of a $\Sigma_1 \cup \cdots \cup \Sigma_p$ for some orthogonal decomposition. Strong Lenz configurations are clearly weak. If S is a weak Lenz configuration, we assume without loss of generality that it is a subset of $\Sigma_1 \cup \cdots \cup \Sigma_p$, and we define $S_i := S \cap \Sigma_i \setminus \{p^+, p^-\}$ ($i = 1, \dots, p$), $S_0 := S \cap \{p^+, p^-\}$, $n_i := |S_i|$ ($i = 0, \dots, p$), $n := |S|$.

5.3.1. Unit distances. Define

$$u_d^L(n) = \max\{u(S) : S \text{ is a weak Lenz configuration of } n \text{ points in } \mathbb{R}^d\}.$$

We call any weak Lenz configuration S of n points in \mathbb{R}^d for which $u(S) = u_d^L(n)$ an *optimised Lenz configuration* (for unit distances). Unlike the even-dimensional case we cannot give an expression for $u_d^L(n)$ more accurate than the estimate $u_d^L(n) = t_p(n) + \Theta(n^{4/3})$ due to Erdős and Pach [16]. However, we next show that an optimised Lenz configuration must be strong for n sufficiently large, depending on d . This implies that $u_d^L(n)$ can be determined if the function $f(n)$, which gives the maximum number of unit distances for n points on a 2-sphere of radius $1/\sqrt{2}$, is known.

Proposition 13. *For each odd $d \geq 7$ there exists $N(d)$ such that all optimised Lenz configurations for unit distances on $n \geq N(d)$ points in \mathbb{R}^d are strong Lenz configurations.*

Proof. Let S be an optimised Lenz configuration on n points. Suppose S is not a strong Lenz configuration. We aim for a contradiction.

Without loss of generality $S_i \setminus C_i \neq \emptyset$ for $i = 1, 2$. Since $u(S_1 \setminus C_1) = O(|S_1 \setminus C_1|^{4/3})$ (Lemma 7(d)) and $S_1 \setminus C_1 \neq \emptyset$, there exists $x \in S_1 \setminus C_1$ with $u(x, S_1 \setminus C_1) = O(|S_1 \setminus C_1|^{1/3}) = O(n^{1/3})$. Also, since $x \neq p^\pm$, $u(x, C_1) \leq 2$. Therefore, $u(x, S_1) = O(n^{1/3})$. Note that for each $i = 2, \dots, p$, x is at distance 1 to all points in $S_i \cap C_i$, but to none of $S_i \setminus C_i$. If we replace x by a new point on C_1 , we lose at most $u(x, S_1)$ unit distances and gain

$\sum_{i=2}^p |S_i \setminus C_i|$. Since $u(S)$ is the maximum over all weak Lenz configurations,

$$\sum_{i=2}^p |S_i \setminus C_i| \leq u(x, S_i) = O(n^{1/3}).$$

By instead considering a point $x \in S_2 \setminus C_2$ we obtain similarly that

$$\sum_{\substack{i=1 \\ i \neq 2}}^p |S_i \setminus C_i| = O(n^{1/3}).$$

Therefore, $|S_i \setminus C_i| = O(n^{1/3})$ for each $i = 1, \dots, p$.

We can now bound $u(S)$ from above. First note that each point of S_0 is at unit distance to all of C_i and none of $\Sigma_i \setminus C_i$, each point of $\Sigma_i \setminus \{p^+, p^-\}$ is at unit distance to at most two points of C_i , and $u(S_i \cap C_i) \leq |S_i \cap C_i|$ (Lemma 7(a)). This gives:

$$\begin{aligned} u(S_i) &\leq u(S_0 \cup S_i) \\ &= u(S_0, S_i) + u(S_i \cap C_i) + u(S_i \cap C_i, S_i \setminus C_i) + u(S_i \setminus C_i) \\ &\leq 2|S_i \cap C_i| + |S_i \cap C_i| + 2|S_i \setminus C_i| + O(|S_i \setminus C_i|^{4/3}) \\ &= O(n) + O((n^{1/3})^{4/3}) = O(n). \end{aligned}$$

Therefore,

$$\begin{aligned} u(S) &\leq t_p(n) + u(S_0 \cup S_1) + \sum_{i=2}^p u(S_i) \\ &= t_p(n) + O(n), \end{aligned}$$

contradicting $u(S) = u_d^L(n) = t_p(n) + \Theta(n^{4/3})$ for large n . \square

5.3.2. *Diameters.* Define

$$M_d^L(n) = \max\{u(S) : S \text{ is a diameter 1 weak Lenz configuration of } n \text{ points in } \mathbb{R}^d\}.$$

We call any diameter 1 weak Lenz configuration S of n points in \mathbb{R}^d for which $u(S) = M_d^L(n)$ an *optimised Lenz configuration* (for diameters).

We show, exactly as the unit distance case, that an optimised Lenz configuration must be strong for large n , and determine the exact value of $M_d^L(n)$.

Proposition 14. *For each odd $d \geq 7$ there exists $N(d)$ such that all optimised Lenz configurations for diameters on $n \geq N(d)$ points in \mathbb{R}^d are strong Lenz configurations. Furthermore,*

$$M_d^L(n) = t_p(n) + \left\lceil \frac{n}{p} \right\rceil + p - 1 = t_p(n - 1) + n - 1 + p.$$

Proof. Choose a set S of n points equally distributed between the orthogonal circles C_1, \dots, C_{p-1} and 2-sphere Σ_p such that the diameter of each $S \cap C_i$ is 1 and furthermore $|S \cap \Sigma_p| = \lceil n/p \rceil$, $|S \cap C_p| = \lceil n/p \rceil - 1$ and $p^+ \in S$. Then clearly $u(S) = t_p(n) + \lceil n/p \rceil + p - 1$. Therefore, $M_d^L(n) \geq t_p(n) + \lceil n/p \rceil + p - 1$. We need this lower bound in a moment.

Now let S be any optimised Lenz configuration on n points. Let $k_i := |S_i \setminus C_i|$ ($i = 1, \dots, p$). We have to show that S is a strong Lenz configuration, i.e., that $k_i = 0$ for all $i = 1, \dots, p$ except at most one.

First consider the case where $S_0 \neq \emptyset$, where without loss of generality, $S_0 = \{p^+\}$. Then

$$\begin{aligned}
u(S) &= u(S \setminus \{p^+\}) + \sum_{i=1}^p u(p^+, S_i) \\
&= \sum_{1 \leq i < j \leq p} u(S_i, S_j) + \sum_{i=1}^p u(S_i) + \sum_{i=1}^p u(p^+, S_i) \\
&= \sum_{1 \leq i < j \leq p} u(S_i, S_j) + \sum_{i=1}^p u(S_i \cup \{p^+\}) \\
&= \sum_{1 \leq i < j \leq p} |S_i||S_j| - \sum_{1 \leq i < j \leq p} k_i k_j + \sum_{i=1}^p u(S_i \cup \{p^+\}) \\
&\leq t_p(n-1) - \sum_{1 \leq i < j \leq p} k_i k_j + \sum_{i=1}^p (n_i + 1) \\
&= t_p(n-1) - \sum_{1 \leq i < j \leq p} k_i k_j + n - 1 + p \\
&= t_p(n) + \left\lceil \frac{n}{p} \right\rceil + p - 1 - \sum_{1 \leq i < j \leq p} k_i k_j.
\end{aligned}$$

Since $u(S) = M_d^L(n) \geq t_p(n) + \lceil n/p \rceil + p - 1$, we obtain $\sum_{1 \leq i < j \leq p} k_i k_j = 0$, which implies that $k_i = 0$ for all i except one. This proves the theorem for the case $S_0 \neq \emptyset$.

Next consider the case where $S_0 = \emptyset$. Without loss of generality $S_1 \setminus C_1 \neq \emptyset$, otherwise $u(S) \leq t_p(n) + p$, a contradiction. By Lemma 7(f), $u(S_1) \leq n_1$. If we remove the points in S_1 and replace them by placing p^+ into S_0 and placing $n_1 - 1$ points of diameter 1 on C_1 to form another set S' of diameter 1, then we lose at most n_1 diameters and gain $n_1 + k_1 \sum_{i=2}^p k_i$. By maximality, $\sum_{i=2}^p k_i = 0$, i.e., the original S was already a strong Lenz configuration and $u(S) = u(S')$. We have already shown that an optimised Lenz configuration that contains p^+ satisfies $u(S') = t_p(n) + \left\lceil \frac{n}{p} \right\rceil + p - 1$. This finishes the case $S_0 = \emptyset$. \square

5.4. The dimension $d = 5$. Consider an orthogonal decomposition $\mathbb{R}^5 = V_0 \oplus V_1 \oplus V_2$ such that $\dim V_0 = 1$ and $\dim V_1 = \dim V_2 = 2$. Choose $r_1 \in (0, 1)$. Let Σ_1 be the 2-sphere in $V_0 \oplus V_1$ with centre o and radius r_1 . Let C_2 be the circle in V_2 with centre o and radius $r_2 := \sqrt{1 - r_1^2}$. Then any point of Σ_1 and any point of C_2 are at unit distance. We call a translate of a finite subset of $\Sigma_1 \cup C_2$ a *strong Lenz configuration* (equivalent to the 5-dimensional ‘‘Lenz configuration’’ of Section 2).

To define a weak Lenz configuration takes more care than for odd $d \geq 7$. Choose an additional parameter $r \in [0, r_1)$ and a point $o' \in V_0$ at distance r to o . Let C_1 be the circle with centre o' and radius $s_1 := \sqrt{r_1^2 - r^2}$ in

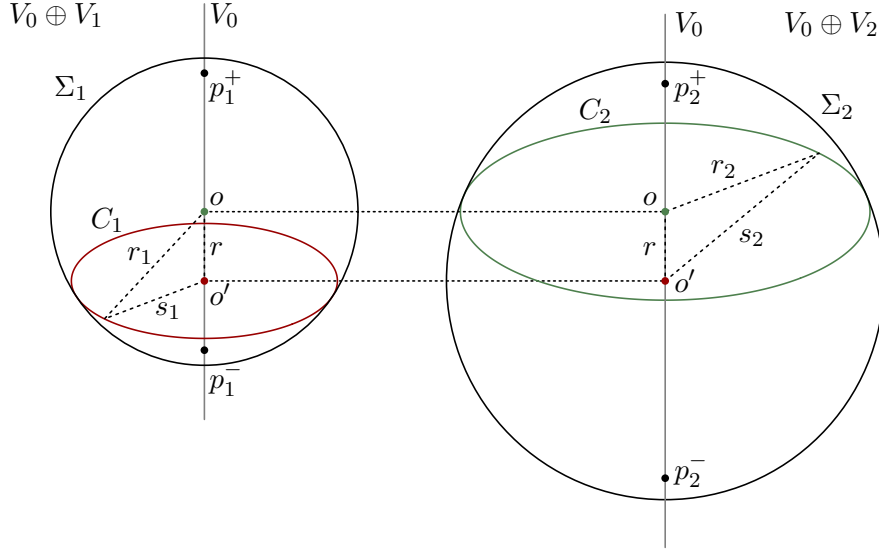


FIGURE 4. Spheres Σ_i and circles C_i of a weak Lenz configuration in \mathbb{R}^5

the plane of $V_0 \oplus V_1$ parallel to V_1 that passes through o' . Let Σ_2 be the 2-sphere in $V_0 \oplus V_2$ with centre o' and radius $s_2 := \sqrt{r_2^2 + r^2}$. Then $C_i \subset \Sigma_i$ ($i = 1, 2$) (Figure 4). Note that $s_1^2 + s_2^2 = 1$, hence any point of Σ_2 and any point of C_1 are at unit distance. Similar to the discussion in Section 5.3 for odd $d \geq 7$, no point of $\Sigma_1 \setminus C_1$ can be at unit distance to a point of $\Sigma_2 \setminus C_2$. We call a translate of a finite subset of $\Sigma_1 \cup \Sigma_2$ a *weak Lenz configuration*. As before, strong Lenz configurations are clearly weak. Assume without loss of generality that $S \subset \Sigma_1 \cup \Sigma_2$. There are also *poles*: $\{p_1^+, p_1^-\} := V_0 \cap \Sigma_1$ and $\{p_2^+, p_2^-\} := V_0 \cap \Sigma_2$. In general, Σ_1 and Σ_2 may not have a point in common. If they do, the common points will be coinciding poles. Define $S_0 := S \cap V_0$, $S_i := S \cap \Sigma_i \setminus V_0$ ($i = 1, 2$) and $n_i := |S_i|$ ($i = 0, 1, 2$), $n := |S|$.

5.4.1. *Unit distances.* Define

$$u_5^L(n) = \max\{u(S) : S \text{ is a weak Lenz configuration of } n \text{ points in } \mathbb{R}^5\}.$$

We call any weak Lenz configuration S of n points in \mathbb{R}^5 satisfying $u(S) = u_5^L(n)$ an *optimised Lenz configuration* (for unit distances). Again the best known estimate is $u_5^L(n) = t_2(n) + \Theta(n^{4/3})$, due to Erdős and Pach [16]. We show that an optimised Lenz configuration is strong for sufficiently large n . As before, this implies that $u_5^L(n)$ can be determined if the function $g(n)$, which gives the maximum number of unit distances for n points on a 2-sphere of arbitrary radius, is known.

Proposition 15. *For all sufficiently large n , all optimised Lenz configurations for unit distances on n points in \mathbb{R}^5 are strong Lenz configurations.*

Proof. Let S be an optimised Lenz configuration on n points. Suppose that $S_1 \setminus C_1 \neq \emptyset$ and $S_2 \setminus C_2 \neq \emptyset$. Then, using Lemma 7(d), there exist points $x_i \in S_i \setminus C_i$ with $u(x_i, S_i \setminus C_i) = O(n^{1/3})$ ($i = 1, 2$). Since $x_i \notin S_0$,

$u(x_i, C_i) \leq 2$. Thus $u(x_i, S_i) = O(n^{1/3})$. If we replace each x_i by a new point on C_i , we lose at most $O(n^{1/3})$ unit distances and gain $|S_1 \setminus C_1| + |S_2 \setminus C_2|$. Since S is extremal, $|S_1 \setminus C_1| + |S_2 \setminus C_2| = O(n^{1/3})$. We bound $u(S)$ from above as in the case of odd $d \geq 7$. For each $i = 1, 2$:

$$\begin{aligned} u(S_i) &\leq u(S_i \cup S_0) \\ &= u(S_0) + u(S_0, S_i) + u(S_i \cap C_i) + u(S_i \cap C_i, S_i \setminus C_i) + u(S_i \setminus C_i) \\ &\leq 4 + 4|S_i \cap C_i| + |S_i \cap C_i| + 2|S_i \setminus C_i| + O(|S_i \setminus C_i|^{4/3}) \\ &= O(n), \end{aligned}$$

hence,

$$\begin{aligned} u(S) &= u(S_1, S_2) + u(S_0 \cup S_1) + u(S_0 \cup S_2) + u(S_0) + u(S_1) + u(S_2) \\ &\leq t_2(n) + O(n), \end{aligned}$$

contradicting $u(S) = t_2(n) + \Theta(n^{4/3})$.

Therefore, some $S_i \setminus C_i = \emptyset$, without loss of generality $S_2 \setminus C_2 = \emptyset$. To show that S is a strong Lenz configuration, it remains to show that $S_0 \subset \Sigma_1$. Suppose then without loss of generality that $p_2^+ \in S_0$ and $p_2^+ \notin \Sigma_1$. Then $p_1^+ \neq p_2^+$. Since p_1^+ is at unit distance to all of C_2 , and p_1^+ and p_2^+ are different points in V_0 , it follows that p_2^+ is not at unit distance to any point in S_2 . If we replace p_2^+ by a new point on C_2 , we lose at most one unit distance (possibly between p_2^+ and p_2^-), and gain $|S \cap \Sigma_1 \setminus C_1|$ unit distances. By extremality, $|S \cap \Sigma_1 \setminus C_1| \leq 1$. Therefore, except for at most 3 points (in addition, $p_2^+ \in S_0$ and possibly $p_2^- \in S_0$), S is on two orthogonal circles, and for this essentially 4-dimensional configuration we obtain $u(S) \leq t_2(n) + O(n)$ as before, a contradiction.

It follows that S is a strong Lenz configuration. \square

5.4.2. *Diameters.* Define

$$M_5^L(n) = \max\{u(S) : S \text{ is a diameter 1 weak Lenz configuration of } n \text{ points in } \mathbb{R}^5\}.$$

We call any diameter 1 weak Lenz configuration S of n points in \mathbb{R}^5 satisfying $u(S) = M_5^L(n)$ an *optimised Lenz configuration* (for diameters). Again an optimised Lenz configuration is strong for large n , and the exact value of $M_5^L(n)$ can be determined. However, this case is more intricate than odd $d \geq 7$.

Proposition 16. *For all sufficiently large n , all optimised Lenz configurations for diameters on n points in \mathbb{R}^5 are strong Lenz configurations. Furthermore, $M_5^L(n) = t_2(n) + n$.*

Proof. We first describe two types of strong Lenz configurations on n points with $t_2(n) + n$ diameters.

In the first construction, choose r_1 such that there exists a set S_1 of n_1 points of diameter 1 on Σ_1 with $2n_1 - 2$ diameters. By Lemma 7(e) this is possible if $n_1 \geq 4$, $n_1 \neq 5$. Choose any set S_2 of $n_2 = n - n_1$ points of diameter 1 on C_2 . (Note that $r_1 < 1/\sqrt{2}$ by Lemma 7(f), which

gives $r_2 > 1/\sqrt{2} > 1/\sqrt{3}$. Then by Lemma 7(c), we can have at most one diameter of length 1 on C_2 .) Let $S := S_1 \cup S_2$. Then

$$\begin{aligned} u(S) &= u(S_1, S_2) + u(S_1) + u(S_2) \\ &= n_1 n_2 + 2n_1 - 2 + 1 = n_1(n_2 + 2) - 1 \\ &\leq t_2(n + 2) - 1 = t_2(n) + n. \end{aligned}$$

Equality is possible by taking $n_1 = \lfloor n/2 \rfloor + 1$ or $\lceil n/2 \rceil + 1$. Keeping in mind that $n_1 \geq 4$, $n_1 \neq 5$, we obtain $t_2(n) + n$ diameters for all $n \geq 6$, $n \neq 8$.

In the second construction, first choose r_2 such that there exists a set S_2 of n_2 points of diameter 1 on C_2 with n_2 diameters (a regular star polygon). By Lemma 7(b) this is possible if $n_2 \geq 3$ is odd. Then $r_2 \leq 1/\sqrt{3}$ by Lemma 7(c), and $r_1 \geq \sqrt{2/3} > 1/\sqrt{2}$. By Lemma 7(f) we can then choose a set S_1 of $n_1 = n - n_2$ points of diameter 1 on Σ_1 with n_1 diameters if $n_1 \geq 3$. Let $S := S_1 \cup S_2$. Then

$$\begin{aligned} u(S) &= u(S_1, S_2) + u(S_1) + u(S_2) \\ &= n_1 n_2 + n_1 + n_2 = (n_1 + 1)(n_2 + 1) - 1 \\ &\leq t_2(n + 2) - 1 = t_2(n) + n. \end{aligned}$$

Equality is possible by taking $n_1 = \lfloor n/2 \rfloor$, $n_2 = \lceil n/2 \rceil$ or $n_1 = \lceil n/2 \rceil$, $n_2 = \lfloor n/2 \rfloor$. Keeping in mind the requirements that $n_2 \geq 3$ must be odd and $n_1 \geq 3$, we obtain $t_2(n) + n$ diameters for all $n \geq 6$, $n \not\equiv 0 \pmod{4}$. (It is because this second, simpler construction does not work for all n that we need the construction in Lemma 7(e) of an odd number n_1 of points on a 2-sphere with $2n_1 - 2$ diameters.)

Summarizing, $M_5^L(n) \geq t_2(n) + n$ for all $n \geq 9$. It is easy to see that all strong Lenz configurations with at least $t_2(n) + n$ diameters must be one of the above two constructions for sufficiently large n . We now turn to weak Lenz configurations.

Let S be an optimised Lenz configuration on n points. We distinguish between two cases.

First case: $S \cap \Sigma_1 \cap \Sigma_2 \neq \emptyset$. Any point in $S \cap \Sigma_1 \cap \Sigma_2$ must be a common pole of Σ_1 and Σ_2 , say $p_1^+ = p_2^+$. Since this point is at distance 1 to C_1 and C_2 , it follows that $|p_1^+ p_1^-|, |p_2^+ p_2^-| > 1$. Therefore, $S \cap \Sigma_1 \cap \Sigma_2$ contains only one point $p := p_1^+ = p_2^+$, at distance 1 to both C_1 and C_2 . Let $k_i := |S_i \setminus C_i|$ ($i = 1, 2$). Then

$$\begin{aligned} t_2(n) + n &\leq u(S) \\ &= u(S_1, S_2) + u(S_1 \cup \{p\}) + u(S_2 \cup \{p\}) \\ &= n_1 n_2 - k_1 k_2 + u(S_1 \cup \{p\}) + u(S_2 \cup \{p\}). \end{aligned} \tag{1}$$

If $u(S_i \cup \{p\}) \leq n_i + 1$ for both $i = 1, 2$, then by substituting into (1),

$$\begin{aligned} t_2(n) + n &\leq n_1 n_2 - k_1 k_2 + n_1 + 1 + n_2 + 1 \\ &= (n_1 + 1)(n_2 + 1) - k_1 k_2 + 1 \\ &\leq t_2(n + 1) - k_1 k_2 + 1 \quad (\text{note } n_1 + n_2 + 1 = n) \\ &= t_2(n) + \left\lceil \frac{n}{2} \right\rceil - k_1 k_2 + 1. \end{aligned}$$

Therefore, $\lfloor n/2 \rfloor + k_1 k_2 \leq 1$, a contradiction.

Without loss of generality we may therefore assume that $u(S_1 \cup \{p\}) > n_1 + 1$. By Lemma 7(f), $r_1 < 1/\sqrt{2}$, which gives $r_2 > 1/\sqrt{2}$ and $u(S_2 \cup \{p\}) \leq n_2 + 1$ (again Lemma 7(f)). Also, $u(S_1 \cup \{p\}) \leq 2(n_1 + 1) - 2 = 2n_1$ (Lemma 7(e)). Substituting into (1),

$$\begin{aligned} t_2(n) + n &\leq n_1 n_2 - k_1 k_2 + 2n_1 + n_2 + 1 \\ &= (n_1 + 1)(n_2 + 2) - k_1 k_2 - 1 \\ &\leq t_2(n + 2) - k_1 k_2 - 1 \\ &= t_2(n) + n - k_1 k_2. \end{aligned}$$

It follows that $k_1 k_2 = 0$, S is a strong Lenz configuration, and $u(S) = t_2(n) + n$.

Second case: $S \cap \Sigma_1 \cap \Sigma_2 = \emptyset$. Then S may still contain poles, but a pole of Σ_i in S is not at distance 1 to C_i (otherwise it would also be a pole of the other sphere). We now define $T_i = S \cap \Sigma_i$ ($i = 1, 2$). Then T_1, T_2 partition S (and we forget about the partition S_0, S_1, S_2). Let $m_i := |T_i|$ and $k_i := |T_i \setminus C_i|$ ($i = 1, 2$). As in the first case,

$$\begin{aligned} t_2(n) + n &\leq u(S) \\ &= u(T_1, T_2) + u(T_1) + u(T_2) \\ &= m_1 m_2 - k_1 k_2 + u(T_1) + u(T_2). \end{aligned} \tag{2}$$

If $u(T_i) \leq m_i$ for both $i = 1, 2$, then by substituting into (2),

$$\begin{aligned} t_2(n) + n &\leq m_1 m_2 - k_1 k_2 + m_1 + m_2 \\ &= (m_1 + 1)(m_2 + 1) - k_1 k_2 - 1 \\ &\leq t_2(n + 2) - k_1 k_2 - 1 \\ &= t_2(n) + n - k_1 k_2. \end{aligned}$$

It follows that $k_1 k_2 = 0$, S is a strong Lenz configuration, and $u(S) = t_2(n) + n$.

Otherwise, without loss of generality, $u(T_1) > m_1$. As in the first case,

$$u(T_1) \leq 2m_1 - 2 \tag{3}$$

and

$$u(T_2) \leq m_2. \tag{4}$$

Since each point in $T_i \setminus C_i$ is joined to at most two points of $T_i \cap C_i$ (recall that in this case a pole is not joined to any point on C_i), we also obtain

$$\begin{aligned} u(T_1) &= u(T_1 \cap C_1) + u(T_1 \cap C_1, T_1 \setminus C_1) + u(T_1 \setminus C_1) \\ &\leq |T_1 \cap C_1| + 2|T_1 \setminus C_1| + 2|T_1 \setminus C_1| - 2 \\ &= m_1 + 3k_1 - 2 \end{aligned} \tag{5}$$

and since $r_2 > 1/\sqrt{2}$,

$$\begin{aligned} u(T_2) &= u(T_2 \cap C_2) + u(T_2 \cap C_2, T_2 \setminus C_2) + u(T_2 \setminus C_2) \\ &\leq 1 + 2|T_2 \setminus C_2| + |T_2 \setminus C_2| \\ &= 1 + 3k_2. \end{aligned} \tag{6}$$

Substituting (4) and (5) into (2):

$$\begin{aligned}
t_2(n) + n &\leq m_1 m_2 - k_1 k_2 + m_1 + 3k_1 - 2 + m_2 \\
&= (m_1 + 1)(m_2 + 1) - k_1(k_2 - 3) - 3 \\
&\leq t_2(n + 2) - k_1(k_2 - 3) - 3 \\
&= t_2(n) + n - k_1(k_2 - 3) - 2.
\end{aligned}$$

Therefore, $k_1(k_2 - 3) + 2 \leq 0$, hence $k_2 \leq 2$.

Substituting (3) and (6) into (2):

$$\begin{aligned}
t_2(n) + n &\leq m_1 m_2 - k_1 k_2 + 2m_1 - 2 + 3k_2 + 1 \\
&= m_1(m_2 + 2) - (k_1 - 3)k_2 - 1 \\
&\leq t_2(n + 2) - (k_1 - 3)k_2 - 1 \\
&= t_2(n) + n - (k_1 - 3)k_2.
\end{aligned}$$

Therefore, $(k_1 - 3)k_2 \leq 0$. If $k_2 > 0$, then $k_1 \leq 3$, and substituting (5) and (6) into (2):

$$\begin{aligned}
t_2(n) + n &\leq m_1 m_2 - k_1 k_2 + m_1 + 3k_1 - 2 + 3k_2 + 1 \\
&= m_1(m_2 + 1) + O(1) \\
&\leq t_2(n + 1) + O(1) \\
&= t_2(n) + \lceil \frac{n}{2} \rceil + O(1),
\end{aligned}$$

a contradiction. It follows that $k_2 = 0$, giving that S is a strong Lenz configuration, and $u(S) = t_2(n) + n$. \square

6. STABILITY THEOREMS

We formulate the stability theorem of Erdős and Simonovits [2, Chapter 5, Theorem 4.2] in the following convenient way. Let $K_r(t)$ denote the complete r -partite graph with t vertices in each class.

Stability Theorem. *For any $p, t \geq 2$ and any $\varepsilon > 0$ there exists N and $\delta > 0$ such that if G is any graph with $n \geq N$ vertices, at least $(\frac{p-1}{2p} - \delta)n^2$ edges and does not contain $K_{p+1}(t)$, then the vertices of G can be partitioned into sets S_0, S_1, \dots, S_p such that $|S_0| < \varepsilon n$, for each $i = 1, \dots, p$,*

$$\frac{n}{p} - \varepsilon n < |S_i| < \frac{n}{p} + \varepsilon n,$$

and each $x \in S_i$ is joined to all vertices of $G - S_i$ with the exception of less than εn .

We now use the Stability Theorem to prove Theorems 4 and 5.

Proof of Theorem 4. Without loss of generality $\varepsilon < 1/(3p^2)$. By Lemma 8, $K_{p+1}(3)$ does not occur in the unit distance graph of S . Let S_0, S_1, \dots, S_p be the partition coming from the Stability Theorem. Suppose S_1 is not on a circle. Let A_1 be a set of 4 nonconyclic points of S_1 . For each $i = 2, \dots, p$, let A_i consist of 3 points of S_i such that any two points in distinct A_i 's are joined. This is possible, since each $x \in S_i$ is at unit distance to all points in $S \setminus S_i$ except for εn points, and $(4 + 3(p - 2))\varepsilon n + 3 < n/p - \varepsilon n$ if $n > 9p^2$. The unit distance graph of $\bigcup_{i=1}^p A_i$ contains a complete p -partite graph with

4 vertices in one class, and 3 vertices in each other class. By Lemma 8, each A_i is concyclic, a contradiction.

Therefore, each S_i ($i = 1, \dots, p$) is concyclic. To see that these circles are orthogonal, choose 3 points from each S_i as above to form a $K_p(3)$. Again by Lemma 8 each class lies on a circle C_i , with C_1, \dots, C_p mutually orthogonal. Since there is a unique circle through any 3 noncollinear points, $S_i \subset C_i$ for each $i = 1, \dots, p$. \square

The following is the even-dimensional case of Corollary 6.

Corollary 17. *Fix an even $d \geq 4$. If a set S of n points in \mathbb{R}^d has at least $(\frac{p-1}{2p} - o(1))n^2$ unit distance pairs, then S is a Lenz configuration except for $o(n)$ points.*

Proof of Theorem 5. Without loss of generality, $\varepsilon < 1/(4p^2)$. By Lemma 8, $K_{p+1}(3)$ does not occur in the unit distance graph of S . Let S_0, S_1, \dots, S_p be the partition coming from the Stability Theorem using $\varepsilon' = \varepsilon/5$. Suppose S_1 is not on a 2-sphere. Let A_1 be a set of 5 points of S_1 that are not contained in any sphere. For each $i = 2, \dots, p$, let A_i consist of 3 points of S_i such that any two points in distinct A_i 's are joined. This is possible, since each $x \in S_i$ is at unit distance to all points in $S \setminus S_i$ except for $\varepsilon'n$ points, and $(5 + 3(p-2))\varepsilon'n + 3 < n/p - \varepsilon'n$ if $n > 4p$. The unit distance graph of $\bigcup_{i=1}^p A_i$ contains a complete p -partite graph with 5 vertices in one class, and 3 vertices in each other class. By Lemma 8, each A_i is on a sphere, a contradiction.

Therefore each S_i ($i = 1, \dots, p$) is on a 2-sphere. If each S_i lies on a circle, then as in the even-dimensional case it follows that these circles are orthogonal. Without loss of generality, S_1 is not concyclic. Let Σ_1 denote the 2-sphere on which S_1 lies. Let A_1 be a set of 4 noncoplanar points of S_1 .

We now modify the partition of S slightly. There are less than $4\varepsilon'n$ points of $\bigcup_{i=2}^p S_i$ not joined to all of A_1 . Remove these points from $\bigcup_{i=2}^p S_i$ and add them to S_0 . Thus we may assume that each point of A_1 is joined to all of $\bigcup_{i=2}^p S_i$, but now we only have $|S_0| < 5\varepsilon'n = \varepsilon n$, for each $i = 1, \dots, p$, $|S_i| - n/p < \varepsilon n$, and each point of S_i is joined to less than εn points of $S \setminus S_i$. We show that for this modified partition, S_2, \dots, S_p are on circles C_2, \dots, C_p , with $\Sigma_1, C_2, \dots, C_p$ mutually orthogonal.

Suppose some S_i ($i = 2, \dots, p$) is not concyclic, without loss of generality S_2 . Let A_2 be 4 nonconcyclic points from S_2 , and as before, for $i = 3, \dots, p$, let A_i be 3 points from S_i such that all points in different A_i 's are joined. By Lemma 8 the A_i lie on spheres in mutually orthogonal subspaces. By choice of A_1 it spans a 3-dimensional space. Since A_2 is cospherical but not concyclic, it also spans a 3-dimensional space. The other A_i each spans at least 2 dimensions. We obtain at least $3 + 3 + 2(p-2) = d + 1$ dimensions, a contradiction.

Therefore, each S_i ($i = 2, \dots, p$) is on a circle C_i . As before, to see that $\Sigma_1, C_2, \dots, C_p$ are mutually orthogonal, choose 4 noncoplanar points from S_1 and 3 points from the other S_i to form a complete p -partite graph, and apply Lemma 8. \square

The following is the odd-dimensional case of Corollary 6.

Corollary 18. *Fix an odd $d \geq 5$. If a set S of n points in \mathbb{R}^d has at least $(\frac{p-1}{2p} - o(1))n^2$ unit distance pairs, then S is a strong Lenz configuration except for $o(n)$ points.*

7. EXTREMAL SETS ARE (WEAK) LENZ CONFIGURATIONS

The following three results, completing the proof of the main theorem, follow relatively simply from the stability theorems.

Proposition 19. *For each even $d \geq 4$ there exists $N(d)$ such that all sets of $n \geq N(d)$ points in \mathbb{R}^d extremal with respect to unit distances or diameters, are Lenz configurations.*

Proof. When considering diameters assume that the diameter is 1. In both cases an extremal set S on n points has at least $\frac{p-1}{2p}n^2$ unit distances, so we may apply Theorem 4 with $\varepsilon = 1/(2p^2)$. Thus for n sufficiently large depending on d we have a partition S_0, S_1, \dots, S_p of S with $|S_0| < \varepsilon n$ and for $i = 1, \dots, p$, $||S_i| - n/p| < \varepsilon n$ and the S_i are on orthogonal circles C_i .

We use the extremality of S to show that $S_0 \subset \bigcup_{i=1}^p C_i$. Let $x \in S_0$. If $u(x, S_i) \geq 3$ for all $i = 2, \dots, p$, then by Lemma 8, x is on a circle of radius $1/\sqrt{2}$ in the plane orthogonal to the span of $\bigcup_{i=2}^p C_i$, i.e., $x \in C_1$. Thus without loss of generality, $u(x, S_i) \leq 2$ for at least two i 's, say $i = 1, 2$. Then

$$\begin{aligned} u(x, S) &= \sum_{i=0}^p u(x, S_i) \leq |S_0| - 1 + 2 + 2 + \sum_{i=3}^p |S_i| \\ &< \varepsilon n - 1 + 4 + (p-2)\left(\frac{n}{p} + \varepsilon n\right) = \left(1 - \frac{2}{p} + \varepsilon(p-1)\right)n + 3. \end{aligned}$$

If we remove x and replace it with a new point $x' \in C_1$, then

$$\begin{aligned} u(x', S \setminus \{x\}) &\geq u(x', \bigcup_{i=2}^p S_i) = \sum_{i=2}^p |S_i| \\ &> (p-1)\left(\frac{n}{p} - \varepsilon n\right) = \left(1 - \frac{1}{p} - (p-1)\varepsilon\right)n. \end{aligned}$$

In the case of diameters we have to take care that x' does not increase the diameter. This can be done as follows.

Since all points of C_1 are already at unit distance to all points of $\bigcup_{i=2}^p C_i$, it is sufficient to choose x' at distance at most 1 to each point of S_0 . When $d \geq 6$, C_1 has radius $1/\sqrt{2}$, hence S_1 is contained in a 90° arc γ of C_1 . The set of points on C_1 at distance larger than 1 from some $y \in S_0$ is a (perhaps empty) subarc of γ . Such a subarc does not contain any point of S_1 , and is therefore between some two consecutive points of S_1 . Since $|S_1| \geq |S_0| + 1$ for n sufficiently large, there exist two consecutive points of S_1 , say a and b , with no subarc between them. Therefore, all points on C_1 between a and b are at distance at most 1 to all points of S_0 , and we may choose x' to be any point on C_1 between a and b .

When $d = 4$, one of the two circles C_1 and C_2 has radius at least $1/\sqrt{2}$, and the above argument also works for this circle.

Since S is extremal, such a modification cannot increase the number of unit distances:

$$u(S) \geq u(S \cup \{x'\} \setminus \{x\}),$$

hence

$$u(x, S) \geq u(x', S \setminus \{x\}),$$

i.e.,

$$\left(1 - \frac{2}{p} + \varepsilon(p-1)\right)n + 3 > \left(1 - \frac{1}{p} - \varepsilon(p-1)\right)n,$$

which is a contradiction if $\varepsilon = 1/(2p^2)$ and $n \geq 3p^2$. Therefore, $x \in C_1$.

We have shown that $S_0 \subset \bigcup_{i=1}^p C_i$, which implies that S is a Lenz configuration for large n . \square

Theorem 20. *For each odd $d \geq 7$ there exists $N(d)$ such that all sets of $n \geq N(d)$ points in \mathbb{R}^d extremal with respect to unit distances or diameters, are weak Lenz configurations.*

Proof. Again in the case of diameters assume that the diameter is 1. An extremal set S on n points has at least $\frac{p-1}{2p}n^2$ unit distances, so we may apply Theorem 5 with $\varepsilon = 1/(4p^2)$. Thus for n sufficiently large depending on d we have a partition S_0, S_1, \dots, S_p of S with $|S_0| < \varepsilon n$ and for $i = 1, \dots, p$, $|S_i| - n/p < \varepsilon n$, S_1 is on a sphere Σ_1 , each S_i ($i = 2, \dots, p$) is on a circle C_i , and $\Sigma_1, C_2, \dots, C_p$ are mutually orthogonal and all have radius $1/\sqrt{2}$.

To show that S is a weak Lenz configuration, it is sufficient to show that each point of S_0 not on Σ_1 lies on the 2-sphere of radius $1/\sqrt{2}$ containing some C_i ($i = 2, \dots, p$) in the subspace generated by C_i and some fixed diameter of Σ_1 .

As in the proof of Theorem 19, extremality of S implies a lower bound on the degree of each point $x \in S$. As before we find a point $x' \in C_2$ without increasing the diameter. Since S is extremal,

$$\begin{aligned} u(x, S) &\geq u(x', S \setminus \{x\}) \geq \sum_{\substack{i=1 \\ i \neq 2}}^p |S_i| \\ &> (p-1) \left(\frac{n}{p} - \varepsilon n \right) = \left(1 - \frac{1}{p} - (p-1)\varepsilon \right) n. \end{aligned} \quad (7)$$

For $i = 2, \dots, p$ define

$$T_i := \{x \in S_0 : u(x, S_i) \leq 2\}.$$

Clearly for any point $x \in \Sigma_1$, $u(x, S_i) = |S_i| > \frac{n}{p} - \varepsilon n \geq 3$ for $n > 4p$, and therefore $\bigcup_{i=2}^p T_i \subseteq S_0 \setminus \Sigma_1$. Conversely, if $x \in S_0$ and $u(x, S_i) \geq 3$ for each $i = 2, \dots, p$, then $x \in \Sigma_1$ (Lemma 8). It follows that $\bigcup_{i=2}^p T_i = S_0 \setminus \Sigma_1$. We next show that T_2, \dots, T_p partition $S_0 \setminus \Sigma_1$. If not, there exists $x \in S_0 \setminus \Sigma_1$

with $u(x, S_i) \leq 2$ and $u(x, S_j) \leq 2$ for distinct $i, j \in \{2, \dots, p\}$. Then

$$\begin{aligned} u(x, S) &= u(x, S_0) + u(x, S_1) + \sum_{i=2}^p u(x, S_i) \\ &< \varepsilon n + \frac{n}{p} + \varepsilon n + 2 + 2 + (p-3) \left(\frac{n}{p} + \varepsilon n \right) \\ &= \left(1 - \frac{2}{p} + (p-1)\varepsilon \right) n + 4, \end{aligned}$$

which contradicts the lower bound (7) when $n > 8p$.

Note that the neighbours in S_1 of an $x \in S_0 \setminus \Sigma_1$ all lie on a circle C_1 , say, of Σ_1 . We now show that this circle is the same for all $x \in S_0 \setminus \Sigma_1$. First we bound $u(x, S_1)$ from below:

$$\begin{aligned} u(x, S) &= u(x, S_0) + u(x, S_1) + \sum_{i=2}^p u(x, S_i) \\ &< \varepsilon n + u(x, S_1) + 2 + (p-2) \left(\frac{n}{p} + \varepsilon n \right) \\ &= u(x, S_1) + \left(1 - \frac{2}{p} + (p-1)\varepsilon \right) n + 2, \end{aligned}$$

which, together with the estimate (7), gives

$$u(x, S_1) > \left(\frac{1}{p} - 2(p-1)\varepsilon \right) n - 2.$$

If the neighbours in S_1 of some other $x' \in S_0 \setminus \Sigma_1$ are on another circle of Σ_1 , then

$$|S_1| \geq u(x, S_1) + u(x', S_1) - 2 > 2 \left(\frac{1}{p} - 2(p-1)\varepsilon \right) n - 6.$$

Since $|S_1| < \frac{n}{p} + \varepsilon n$, we have a contradiction if $n > 8p^2$.

Therefore, the neighbours in S_1 of any $x \in S_0 \setminus \Sigma_1$ are on C_1 . Since C_1 contains at least 3 points of S_1 , it is orthogonal to C_2, \dots, C_p (Lemma 8), and therefore it has radius $1/\sqrt{2}$, and is a great circle of Σ_1 . For each $i = 2, \dots, p$, let Σ_i be the sphere of radius $1/\sqrt{2}$ which has C_i as great circle, in the 3-space containing C_i and the diameter of Σ_1 perpendicular to C_1 . Since T_2, \dots, T_p is a partition, each point of T_i is at distance 1 to at least 3 points of each C_j , $j \neq i$, and by Lemma 8, $T_i \subset \Sigma_i$. Since also $S_i \subset \Sigma_i$, we have shown that S is a weak Lenz configuration for large n . \square

Theorem 21. *For all sufficiently large n , all sets of n points in \mathbb{R}^5 extremal with respect to unit distances or diameters are weak Lenz configurations.*

Proof. An extremal set S of n points has at least $n^2/4$ unit distances, so by Theorem 5 with $\varepsilon = 1/11$ we obtain that for sufficiently large n , S can be partitioned into S_0, S_1, S_2 such that $|S_0| < \varepsilon n$, $||S_i| - n/2| < \varepsilon n$ ($i = 1, 2$), S_1 is on a sphere Σ_1 of radius r_1 , S_2 is on a circle C_2 of radius r_2 , such that Σ_1 and C_2 are orthogonal and $r_1^2 + r_2^2 = 1$.

As in the proof for odd $d \geq 7$, if $r_2 \geq 1/\sqrt{2}$, we can find a point $x' \in C_2$ that does not increase the diameter. Otherwise, $r_1 \geq 1/\sqrt{2}$, and we consider

the intersection of Σ_1 and all balls in the 3-space of Σ_1 of radius 1 centred at points in $S \cap \Sigma_1$. This gives a spherically convex set on Σ_1 containing $S \cap \Sigma_1$. Any new point x' in this set is at distance at most 1 to all points of S . As before, replacing any point $x \in S$ by x' gives $u(x, S) > (\frac{1}{2} - \varepsilon)n$. Note that if $u(x, S_2) \geq 3$ for some $x \in S_0$, then $x \in \Sigma_1$. Therefore, $u(x, S_2) \leq 2$ for all $x \in S_0 \setminus \Sigma_1$. Next we bound $u(x, S_1)$ from below for all $x \in S_0 \setminus \Sigma_1$:

$$\begin{aligned} \left(\frac{1}{2} - \varepsilon\right)n &< u(x, S) = u(x, S_0) + u(x, S_1) + u(x, S_2) \\ &< \varepsilon n + u(x, S_1) + 2, \end{aligned}$$

hence

$$u(x, S_1) > \left(\frac{1}{2} - 2\varepsilon\right)n - 2.$$

The neighbours in S_1 of an $x \in S_0 \setminus \Sigma_1$ lie on a circle C_1 , say, of Σ_1 . If the neighbours of some other $x' \in S_0 \setminus \Sigma_1$ lie on another circle of Σ_1 , then

$$\begin{aligned} \frac{n}{2} + \varepsilon n &> |S_1| > u(x, S_1) + u(x', S_1) - 2 \\ &> (1 - 4\varepsilon)n - 6. \end{aligned}$$

Therefore, $5\varepsilon n > \frac{n}{2} - 6$, a contradiction for n sufficiently large.

Let the radius of C_1 be s_1 . By Lemma 8, each $x \in S_0 \setminus \Sigma_1$ lies on its complementary sphere Σ_2 of radius s_2 , where $s_1^2 + s_2^2 = 1$, and $C_2 \subset \Sigma_2$. We have shown that S is a weak Lenz configuration for large n . \square

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